1. $\frac{\partial f}{\partial x} = e^y$, $\frac{\partial f}{\partial y} = xe^y + \sin z$, $\frac{\partial f}{\partial z} = y\cos z$

$\frac{\partial f}{\partial x}(2, \ln 3, \frac{\pi}{2}) = 3$, $\frac{\partial f}{\partial y}(2, \ln 3, \frac{\pi}{2}) = 7$

$\frac{\partial f}{\partial z}(2, \ln 3, \frac{\pi}{2}) = 0$

By the formula $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$,

the maximum possible error is approximately

$$3 \cdot 0.2 + 7 \cdot 0.6 = 4.8$$

2. $\frac{\partial f}{\partial x}(a,b) = y\cos x \Big|_{(a,b)} = b\cos a$

$\frac{\partial f}{\partial y}(a,b) = \sin x \Big|_{(a,b)} = \sin a$

The total derivative $f'(a,b)$ is $(b\cos a, \sin a)$

Verification: let $\langle h_1, h_2 \rangle \neq (0,0)$

$f(a+h_1, b+h_2) - f(a,b) - f'(a,b)\langle h_1, h_2 \rangle$

$= (b+h_2)\sin(a+h_1) - b\sin a - h_1 b\cos a - h_2 \sin a$

$= (b\sin(a+h_1) - b\sin a - h_1 b\cos a) + h_2(\sin(a+h_1) - \sin a)$

$$\therefore \left.\frac{d}{dx}\sin x\right|_{x=a} = \cos a$$

$$= b\big(\underbrace{\sin(a+h_1) - \sin a - h_1\cos a}_{= o(|h_1|)}\big) + h_2(\sin(a+h_1) - \sin a)$$

$$= o(|h_1|) + h_2(\sin(a+h_1) - \sin a)$$

- We want to check that

$$h_2(\sin(a+h_1) - \sin a) = o(\sqrt{h_1^2+h_2^2}) \text{ as } \sqrt{h_1^2+h_2^2} \to 0$$

Note that

① $\because \sqrt{h_1^2+h_2^2} \geq |h_2| \quad \therefore \left|\frac{h_2}{\sqrt{h_1^2+h_2^2}}\right| \leq 1$

② $|\sin(a+h_1) - \sin(a)| \longrightarrow 0 \text{ as } h_1 \to 0$

③ $\because \sqrt{h_1^2+h_2^2} \geq |h_1| \quad \therefore h_1 \to 0$ as

$\sqrt{h_1^2+h_2^2} \longrightarrow 0$

$\therefore$ By Sandwich theorem,

$$\left|\frac{h_2}{\sqrt{h_1^2+h_2^2}}(\sin(a+h_1) - \sin a)\right| \longrightarrow 0 \text{ as}$$

$\sqrt{h_1^2+h_2^2} \to 0$

By definition,

$$h_2(\sin(a+h_1) - \sin a) = o(\sqrt{h_1^2+h_2^2}) \text{ as } \sqrt{h_1^2+h_2^2} \to 0$$

$$\therefore\ f(a+h_1, b+h_2) - f(a,b) - f'(a,b)\langle h_1, h_2\rangle$$

$$= o(|h_1|) + o(\sqrt{h_1^2+h_2^2})$$

$$= o(\sqrt{h_1^2+h_2^2})$$

$\because$ $\sqrt{h_1^2+h_2^2} \geq |h_1|$, any function that is $o(|h_1|)$ is $o(\sqrt{h_1^2+h_2^2})$.

3(a) If we adopt the convention that the y-axis points the North direction and the x-axis points the East direction, then the North east direction is

$$\vec{u} = \langle \cos\tfrac{\pi}{4}, \sin\tfrac{\pi}{4}\rangle = \langle \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\rangle$$

$$\nabla z(-100,-100) = \left\langle \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right\rangle \bigg|_{(x,y)=(-100,-100)}$$

$$= \langle 0.6, 0.8\rangle$$

$$\frac{\partial z}{\partial \vec{u}}\bigg|_{(-100,-100)} = \langle 0.6, 0.8\rangle \cdot \langle \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\rangle$$

$$= 0.7\sqrt{2} \approx 0.9899$$

(b) The direction vector of $\nabla g(-100, -100)$ is $\langle \frac{3}{5}, \frac{4}{5} \rangle$. Therefore the direction for the minimum rate of rise is $\langle -\frac{3}{5}, -\frac{4}{5} \rangle$ and the rate is

$$\langle 0.6, 0.8 \rangle \cdot \langle \frac{-3}{5}, \frac{-4}{5} \rangle = -1$$

4. The total derivative $\vec{f}'(\vec{a})$ is a $m \times n$ matrix (or a linear transformation : $\mathbb{R}^n \to \mathbb{R}^m$) so that

$$\frac{|\vec{f}(\vec{a}+\vec{h}) - \vec{f}(\vec{a}) - \vec{f}'(\vec{a})\vec{h}|}{|\vec{h}|} \to 0 \quad \text{as } |\vec{h}| \to 0$$

Remarks :

① $\vec{f}'(\vec{a}) \cdot \vec{h}$ is a matrix multiplication

$\vec{f}'(\vec{a})$ multiplied by $\vec{h}$

② $|\vec{f}(\vec{a}+\vec{h}) - \vec{f}(\vec{a}) - \vec{f}'(\vec{a})\vec{h}|$ is the magnitude of a vector in $\mathbb{R}^m$

③ $|\vec{h}|$ is the magnitude of a vector in $\mathbb{R}^n$

④ If we write out in the matrix form,

then

$$\vec{f}(\vec{a}+\vec{h}) - f(\vec{a}) - \vec{f}'(\vec{a})\vec{h} = \begin{pmatrix} \delta_1(\vec{h}) \\ \delta_2(\vec{h}) \\ \vdots \\ \delta_m(\vec{h}) \end{pmatrix}$$

where $\delta_1, \delta_2, \ldots, \delta_m$ are real-valued functions

The requirement for $\vec{f}'(a)$ is that

$$\delta_i(\vec{h}) = o(|\vec{h}|) \quad \text{for each } i = 1, 2, \ldots, m$$

If such a matrix $\vec{f}'(\vec{a})$ exists, then

$$\vec{f}'(\vec{a}) = \begin{pmatrix} \text{---} \ \nabla f_1(\vec{a}) \ \text{---} \\ \text{---} \ \nabla f_2(\vec{a}) \ \text{---} \\ \vdots \\ \text{---} \ \nabla f_m(\vec{a}) \ \text{---} \end{pmatrix}$$

the ith row of $\vec{f}'(\vec{a})$ is $\nabla f_i(\vec{a})$

and the ij th entry $[\vec{f}'(\vec{a})]_{ij} = \frac{\partial f_i}{\partial x_j}(\vec{a})$

5. Objective function : $f(x,y) = x^2 + y^2$

Constraint : $g(x,y) = x^2 + xy + y^2 - 3 = 0$

Using the method of the Lagrange mult.

we look for (x,y) so that

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) & \text{for some } \lambda \in \mathbb{R} \\ g(x,y) = 0 \\ \nabla g(x,y) \neq 0 \end{cases}$$

i.e. $$\begin{cases} (2x, 2y) = \lambda (2x+y, x+2y) & \text{— ①} \\ x^2 + xy + y^2 = 3 & \text{— ②} \\ (2x+y, x+2y) \neq (0,0) & \text{— ③} \end{cases}$$

Note that by the hint,

$$\begin{cases} 2x + y = 0 \\ x + 2y = 0 \end{cases} \quad \text{iff} \quad (x,y) = (0,0)$$

But ② $\Rightarrow$ this is impossible

$\therefore$ We first conclude that $\nabla g(x,y) \neq (0,0)$

for every point (x,y) on the rotated ellipse.

From ①, we have $$\begin{cases} (2-2\lambda) x - \lambda y = 0 \\ -\lambda x + (2-2\lambda) y = 0 \end{cases}$$

By ② again, we need some nontrivial sol^n for the system of linear equations

i.e. $4(1-\lambda)^2 = \lambda^2$

$\Leftrightarrow \quad 2(1-\lambda) = \lambda \quad$ or $\quad 2(1-\lambda) = -\lambda$

$\Leftrightarrow \quad \lambda = \frac{2}{3} \quad$ or $\quad \lambda = 2$

For $\lambda = \frac{2}{3}$, we have

$$\begin{cases} x - y = 0 \\ x^2 + xy + y^2 = 3 \end{cases}$$

Solving $\Rightarrow \quad x^2 + x^2 + x^2 = 3$

$$x^2 = 1$$

That is $(x,y) = (1,1)$ or $(-1,-1)$

For $\lambda = 2$, we have

$$\begin{cases} x + y = 0 \\ x^2 + xy + y^2 = 3 \end{cases}$$

$$\Leftrightarrow \begin{cases} x + y = 0 \\ x^2 = 3 \end{cases}$$

$\Leftrightarrow \quad (x,y) = (\sqrt{3}, -\sqrt{3})$ or $(-\sqrt{3}, \sqrt{3})$

It suffices to check all these three points

$(1,1)$, $(-1,1)$, $(\sqrt{3},-\sqrt{3})$, $(-\sqrt{3},\sqrt{3})$,

$f(1,1)=f(-1,-1)=2$, $f(\sqrt{3},-\sqrt{3})=f(-\sqrt{3},\sqrt{3})=6$

$\therefore$ The points that are nearest to the origin are $(-1,-1)$, $(1,1)$, farthest from the origin are $(\sqrt{3},-\sqrt{3})$ and $(-\sqrt{3},\sqrt{3})$

6. Objective function : $f(x,y,z)=x^2+y^2+z^2$

Constraints:

$$\begin{cases} g(x,y,z) = x+y+z-1 = 0 \\ h(x,y,z) = x^2+y^2-1 = 0 \end{cases}$$

Using Lagrange multiplier, aim to find (x,y,z) so that

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z)$$

$$(2x,2y,2z) = \lambda(1,1,1) + \mu(2x,2y,0) \quad \text{— (1)}$$

for some $\lambda, \mu \in \mathbb{R}$

Notice that the assumption of the method of the Lagrange multiplier (the vector $\nabla g(x,y,z)$ is $\underline{\boxed{\text{NOT}}}$ parallel to $\nabla h(x,y,z)$) is satisfied whenever $(x,y) \neq (0,0)$. In case $(x,y) = (0,0)$, the constraint $h(x,y,z)=0$ fails.

Therefore every point (x,y,z) satisfying the constraints will satisfy the assumption of the method of the Lagrange multipliers automatically.

By ①,

$$\begin{cases} \lambda = 2z \\ 2y = 2z + 2\mu y \\ 2x = 2z + 2\mu x \end{cases}$$

i.e. $z = (1-\mu) y = (1-\mu) x$

Suppose $\mu = 1$, then $z = 0$

Substituting back to the constraint functions, we have $\begin{cases} x+y=1 & — ② \\ x^2+y^2=1 & — ③ \end{cases}$

$②^2 - ③$: $2xy = 0$

$x=0$ or $y=0$

$\therefore$ All possible solutions are

$(0,1,0)$ and $(1,0,0)$ for $\mu=1$

For $\mu \neq 1$, we have $x=y$

The constraints force $(x,y) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ or $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

Then, $(x,y,z) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1-\sqrt{2})$ or $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 1+\sqrt{2})$

Checking these four points

$A = (0,1,0)$, $B = (1,0,0)$, $C = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1-\sqrt{2})$

$D = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 1+\sqrt{2})$.

$f(A) = 1$, $f(B) = 1$. $f(C) = 3-2\sqrt{2}$

$f(D) = 3+2\sqrt{2}$

$\therefore$ The points nearest to the origin are

$(0,1,0)$ and $(1,0,0)$

The point farthest away from the origin is

$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 1+\sqrt{2}\right)$

7. Objective function $f(x,y,z) = xyz$

(The volume of the box is $4xyz$)

Constraints :

$$g(x,y,z) = z - 48 + 3x^2 + 4y^2 = 0$$

$0 \leq x \leq 4$

$$\left(\begin{aligned} 3x^2 &= 48 - z - 4y^2 \\ &\leq 48 \quad [\because z \geq 0] \\ \therefore\ x &\leq 4 \end{aligned}\right)$$

$0 \leq y \leq \frac{\sqrt{48-3x^2}}{2}$

$$\left(\begin{aligned} 4y^2 &= 48 - z - 3x^2 \\ &\leq 48 - 3x^2 \quad [\because z \geq 0] \\ y &\leq \frac{\sqrt{48-3x^2}}{2} \end{aligned}\right)$$

$$\nabla f = (yz, xz, xy)$$

$$\nabla g = (6x, 8y, 1) \neq 0$$

$$\nabla f = \lambda \nabla g \Rightarrow \frac{yz}{6x} = \frac{xz}{8y} = \frac{xy}{1} = \lambda$$

if $x, y \neq 0$

Then, $6x^2 = z$ and $8y^2 = z$

$g(x,y,z) = 0 \Rightarrow z = 24$

$\therefore (x,y,z) = (2, \sqrt{3}, 24)$

① If x or $y=0$, then $xyz=0$

② On the boundary $x=0$, $y=0$, or $z=0$,

we have $xyz=0$

$\therefore$ The maximum possible volume of the box is $4 \cdot (2)(\sqrt{3})(24) = 192\sqrt{3}$

(c.f. Lecture notes 8 p.92)

8. The case $x=y=z=0$ is easy to see.

Assume either one of them is nonzero.

Let $\alpha = \dfrac{1}{x+y+z}$

and put $u = \sqrt{\dfrac{x}{x+y+z}}$, $v = \sqrt{\dfrac{y}{x+y+z}}$

$w = \sqrt{\dfrac{z}{x+y+z}}$. Then, $u^2 = \alpha x$, $v^2 = \alpha y$, $w^2 = \alpha z$

$$xyz = \left(\frac{1}{\alpha}\right)^3 u^2 v^2 w^2$$

$$x+y+z = \frac{1}{\alpha}(u^2+v^2+w^2) = \frac{1}{\alpha}$$

$\therefore \quad \sqrt[3]{xyz} \le \frac{1}{3}(x+y+z)$

$\Leftrightarrow \quad (u^2v^2w^2)^{\frac{1}{3}} \le \frac{1}{3}$

where (u,v,w) is some point on the unit sphere. That is $u^2+v^2+w^2=1$

It is now formulated as the assertion that the maximum value of

$$f(u,v,w) = (u^2v^2w^2)^{\frac{1}{3}}$$

on the unit sphere $u^2+v^2+w^2=1$ is not larger than $\frac{1}{3}$

Objective fcn : $f(u,v,w) = (u^2v^2w^2)^{\frac{1}{3}}$

Constraint : $g(u,v,w) = u^2+v^2+w^2-1=0$

$$\nabla f = \left(\frac{2}{3}u^{-\frac{1}{3}}, \frac{2}{3}v^{-\frac{1}{3}}, \frac{2}{3}w^{-\frac{1}{3}}\right)$$

$$\nabla g = (2u, 2v, 2w) \quad (\neq 0 \text{ when } g(u,v,w)=0)$$

No problem because you don't expect

The maximum value of f is attained when $u=0$, $v=0$ or $w=0$

$$\nabla f = \lambda \nabla g$$

$$\Rightarrow \frac{2u}{\frac{2}{3}u^{-\frac{1}{3}}} = \frac{2v}{\frac{2}{3}v^{-\frac{1}{3}}} = \frac{2w}{\frac{2}{3}w^{-\frac{1}{3}}} = \frac{1}{\lambda}$$

$$\Rightarrow u^{\frac{4}{3}} = v^{\frac{4}{3}} = w^{\frac{4}{3}}$$

$$\Rightarrow u^2 = v^2 = w^2 = \frac{1}{3} \quad (\because g(u,v,w) = 0)$$

$$\therefore \max_{u^2+v^2+w^2=1} f(u,v,w) = \frac{1}{3}$$